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Tidal Heating Section
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Modelling tidal dissipation in Io

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Abstract. Jupiter's moon Io is of particular interest due to its unusual heat source. By far the most geologically active body in the solar system, it has a surface heat flux hundreds of times higher than expected from radiogenic heating. Orbital resonance with the other Galilean satellites causes a forced eccentricity in Io's orbit, and as a result of the continuously varying distance from Jupiter the extent of Io's tidal deformation changes throughout each orbital period. The internal friction caused by this warping generates a large amount of heat. In this paper we first investigate the exact nature and time-dependence of Io's tidal deformation by assuming a fluid model, and then using principles of harmonic oscillation we calculate the approximate heat produced as a function of tidal phase angle, subsequently making an estimate for the surface temperature of Io. Our model will also include a qualitative discussion of Io's interior and of implications for the future of the Jovian system.

I. INTRODUCTION

Io is the innermost of the four largest Jovian moons that are known as the Galilean satellites. It is the most geologically active body in the solar system, with hundreds of volcanoes on its surface and a vast magma ocean beneath its thin crust. Indeed, Io has an observed average surface heat flux of between 1 and $2\,\mathrm{J\,m^{-2}}$ [1, 2], compared to Earth's $0.06\,\mathrm{J\,m^{-2}}$. Unlike most natural satellites in the solar system, whose internal heating comes mostly from radiogenic decay, Io's primary source of power comes from the changing tidal forces that act on the body during its elliptical orbit around Jupiter.

Although Io's orbit has a very low free eccentricity of 0.00001 [3], the orbital resonance of Io, Europa and Ganymede (cf. section VI) causes a forced eccentricity of $e = 0.0041$. As a result, Io's distance from Jupiter varies from 4.200×10^8 m at perijove to 4.234×10^8 m at apojove, and so the magnitude of the tidal force exerted on Io by Jupiter oscillates, varying Io's tidal deformation.

The viscoelastic interior of Io generates a resistive force to the tidal movement, and this frictional force dissipates energy through heat as the tides cycle, a process which would circularise its orbit were it not for its orbital resonance with Europa and Ganymede. It is this heat which is responsible for the enormous energy flux observed.

We will start by deriving exactly how Io deforms in Jupiter's gravitational field. After making several approximations, this will allow us to reach an estimate for the general order of magnitude of the heat produced.

Io's tidal heating has been studied in various ways in existing literature; perhaps most notable is the work done by Segatz *et al.* [4] to model Io's interior and the distribution of tidal dissipation rate across the surface, focusing on multilayer Maxwell rheology models. Moore [5] did extensive work on convection currents within Io responsible for allowing the heat produced to flow to the surface. Yoder [6] also did research into the effects of tidal heating on the formation of resonance locks with the other Galilean satellites. These former two papers focus closely on the specifics of Io's interior structure and its effects on the exact nature and distribution of tidal heating. The latter paper focuses on context in the Jovian system. While we will touch on both of these topics, we are primarily interested in a phenomenological model that can be used to predict the approximate nature of Io's behaviour given only common empirical values. This will generate a model that can more easily be transferred to other planetary systems, and as such our approach is unique.

II. DEFORMATION OF IO IN THE GRAVITATIONAL FIELD OF JUPITER

A. Finding the deformed shape of Io

We want to start by considering all of the forces acting on any point on the surface of Io. Let L be the distance between the centres of Io and Jupiter, and let l be the distance between the centre of Io and the barycentre of orbit. Clearly

$$\begin{aligned} l &= \frac{M_J}{M_I + M_J} L \\ &= 0.999953\, L \end{aligned}$$

where M_J is the mass of Jupiter and M_I is the mass of Io, and so we can make the approximation $l \approx L$. That is, we can take Io's orbital barycentre to be the centre of Jupiter.

The spherical polar coordinate system that we'll use is shown in figs. 1 and 2; taking Io's centre as the origin, r is the distance to an arbitrary point on Io's surface, θ is the polar angle (latitude) and ϕ is the azimuth angle (longitude).

Consider the non-inertial frame of reference in which we are at a point directly above the centre of Jupiter in line with the axis of rotation and rotating with angular speed ω such that the Jupiter-Io system appears stationary (ignoring the bodies' varying separation). Note that because of how Io is tidally locked,

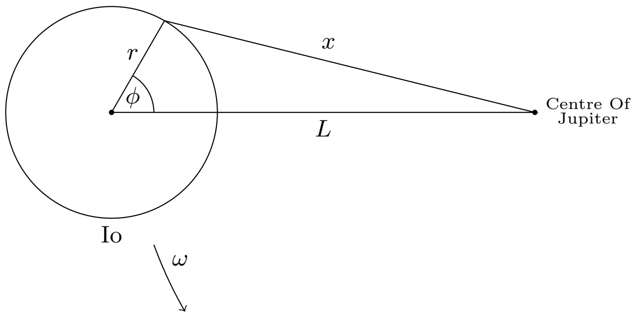

FIG. 1. A cross-sectional view in the plane perpendicular to the axis of rotation and through the centres of Io and Jupiter.

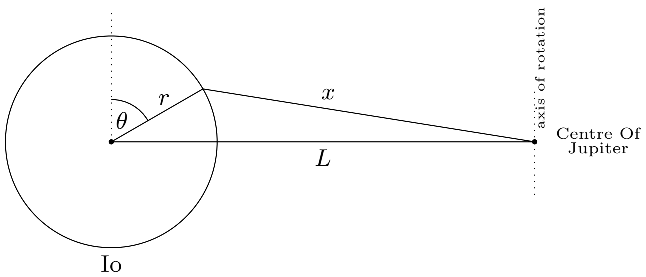

FIG. 2. A cross-sectional view in the plane containing the axis of rotation and through the centres of Io and Jupiter.

in this frame of reference the surface of Io will not appear to move either.

At any point (r, θ, ϕ) on Io's surface, let x be the distance from that point to the centre of Jupiter. The potential energy of a point mass m at coordinates (r, θ, ϕ) consists of three parts:

1. Gravitational potential energy from Io, given by
$$V_I = -G\frac{mM_I}{r}. \tag{1}$$
2. Gravitational potential energy from Jupiter, given by
$$V_J = -G\frac{mM_J}{x}. \tag{2}$$
3. Centrifugal potential energy arising from the fictitious centrifugal force. Since potential energy is given by
$$\begin{aligned} V'(x) &= -F(x), \\ \implies V(x) &= \int -F(x) \cdot \mathrm{d}x, \end{aligned}$$
we can calculate the centrifugal potential energy to be
$$V_C = \int_0^{x_p} -\omega^2 s \,\mathrm{d}s = -\frac{1}{2}m\omega^2 x_p^2$$
where ω is the angular velocity of orbit and x_p is the component of x in the plane of rotation. (Clearly the centrifugal force only depends upon this quantity.) Where h is the perpendicular height above the Io-Jupiter axis,
$$\begin{aligned} x_p &= \sqrt{x^2 - h^2} \\ &\approx x \qquad \text{as}\, h \ll x. \end{aligned}$$
and so the centrifugal potential energy is given by
$$V_C = -\frac{1}{2}m\omega^2 x^2. \tag{3}$$

Therefore, by combining eqs. (1) to (3), we see that the total potential energy V of a mass m is given by
$$\frac{V}{m} = -\frac{1}{2}\omega^2 x^2 - G\frac{M_J}{x} - G\frac{M_I}{r}. \tag{4}$$

Now let α be the angle between (r, θ, ϕ) and $(L, \frac{\pi}{2}, 0)$ a.k.a. Jupiter. By looking at fig. 3, we see that
$$\begin{aligned} \cos\alpha &= \frac{r\sin\theta\cos\phi}{r} \\ &= \sin\theta\cos\phi. \end{aligned}$$

Now, by the cosine rule, the distance x to Jupiter is

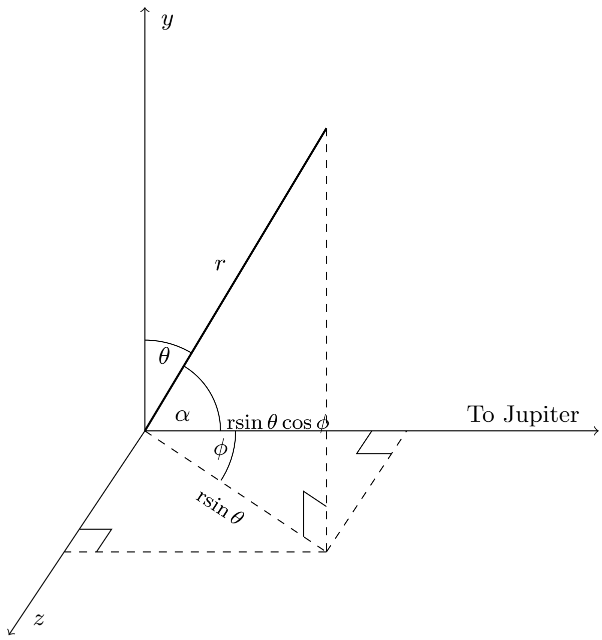

FIG. 3. Our new angle α between (r, θ, ϕ) and Jupiter is shown.

given by
$$\begin{aligned} x^2 &= L^2 + r^2 - 2rL\cos\alpha \\ \implies x &= \sqrt{L^2 + r^2 - 2rL\cos\alpha} \\ &= L\sqrt{1 + \left(\frac{r}{L}\right)^2 - 2\frac{r}{L}\cos\alpha}. \end{aligned}$$

Using the Taylor expansion

$$\frac{1}{\sqrt{1+a^2-2ab}}=1+ab+\frac{1}{2}(3b^2-1)a^2+\frac{1}{2}b(5b^2-3)a^3+\cdots$$

where b is kept constant, we come to the approximation valid for $a \ll 1$ of

$$\frac{1}{\sqrt{1+a^2-2ab}}\approx 1+ab+\frac{1}{2}a^2(3b^2-1)$$

and so since $r \ll R$, we can approximate

$$\frac{1}{x}=\frac{1}{L}\left(1+\frac{r}{L}\cos\alpha+\frac{1}{2}\frac{r^2}{L^2}(3\cos^2\alpha-1)\right).$$

Substituting this into eq. (4) gives us

$$\begin{aligned}\frac{V}{m}&=-\frac{1}{2}\omega^2(L^2+r^2-2rL\cos\alpha)-\frac{GM_J}{L}\left(1+\frac{r}{L}\cos\alpha+\frac{1}{2}\frac{r^2}{L^2}(3\cos^2\alpha-1)\right)-G\frac{M_I}{r}\\&=-\frac{1}{2}\omega^2r^2-G\frac{M_I}{r}-\frac{GM_Jr^2}{2L^3}(3\cos^2\alpha-1)+\omega^2rL\cos\alpha-GM_J\frac{r}{L^2}\cos\alpha-\frac{1}{2}\omega^2L^2-\frac{GM_J}{L}.\end{aligned}\tag{5}$$

Since by considering circular motion on Io as a body we know

$$\omega^2=\frac{GM_J}{L^3},\tag{6}$$

we have

$$\begin{aligned}\omega^2rL\cos\alpha-GM_J\frac{r}{L^2}\cos\alpha&=r\cos\alpha\left[L\cdot\frac{GM_J}{L^3}-\frac{GM_J}{L^2}\right]\\&=0,\end{aligned}$$

and since ω, L, G and M_J are all constants, $-\frac{1}{2}\omega^2L^2-\frac{GM_J}{L}$ is a constant. Hence, eq. (5) becomes

$$\frac{V}{m}=-\frac{1}{2}\omega^2r^2-G\frac{M_I}{r}-G\frac{M_Jr^2}{2L^3}(3\cos^2\alpha-1)+C\tag{7}$$

for some constant C.

Now in order to quantitatively model the shape that Io forms we will first assume it to be a fluid such that the mass will adjust hydrostatically to form an equipotential surface. The effect on the final deformation should be mainly a matter of amplitude as we assume the shape made by both a fluid body and solid body will be of the same topology. If we so wish, we can later multiply the fluid height by a scaling factor in order to reach an approximate value for the tide at any point on the surface of Io.

Where R is the mean radius of Io, the fluid tidal height is therefore $h_f=r-R$. Given that $h_f \ll R$,

$$\begin{aligned}\frac{1}{r}&=\frac{1}{R+h_f}\\&=\frac{1}{R}\cdot\frac{1}{1+\frac{h_f}{R}}\\&\approx\frac{1}{R}\left(1-\frac{h_f}{R}\right)\\&=\frac{1}{R}-\frac{h_f}{R^2}\end{aligned}\tag{8}$$

and

$$\begin{aligned}r^2&=R^2+2Rh_f+h_f^2\\&\approx R^2+2rh_f.\end{aligned}\tag{9}$$

Combining eqs. (6), (8) and (9) into eq. (7),

$$\begin{aligned}\frac{V}{m}=&-\frac{G(M_I+M_J)R}{L^3}h_f+\frac{GM_I}{R^2}h_f\\&-\frac{GM_Jr^2}{2L^3}(3\cos^2\alpha-1)+C_2\end{aligned}$$

for some other constant C_2. Since the ratio of the first term to the second terms is

$$\frac{M_I+M_J}{M_I}\frac{R^3}{L^3}\approx 10^{-3},$$

we consider its contribution negligible and so we can now say

$$\frac{V}{m}=\frac{GM_I}{R^2}h_f-\frac{GM_Jr^2}{2L^3}(3\cos^2\alpha-1)+C_2.$$

Because we are modelling the surface as equipotential, V must be a constant at any point a distance r from Io's centre. This means that the first two terms must compensate each other and so

$$\begin{aligned}\frac{GM_I}{R^2}h_f&=\frac{GM_Jr^2}{2L^3}(3\cos^2\alpha-1)\\\implies h_f&=\frac{M_J}{M_I}\cdot\frac{R^2r^2}{2L^3}(3\cos^2\alpha-1).\end{aligned}$$

Again, given $R \gg h_f$, we can now approximate $r^2\approx R^2$. Therefore, our fluid tidal height is

$$h_f=\frac{M_J}{M_I}\frac{R^4}{2L^3}(3\cos^2\alpha-1).\tag{10}$$

This situation is rotationally symmetric around the Jupiter-Io axis as one might expect. A polar plot of Io's tidal deformation is shown in fig. 4.

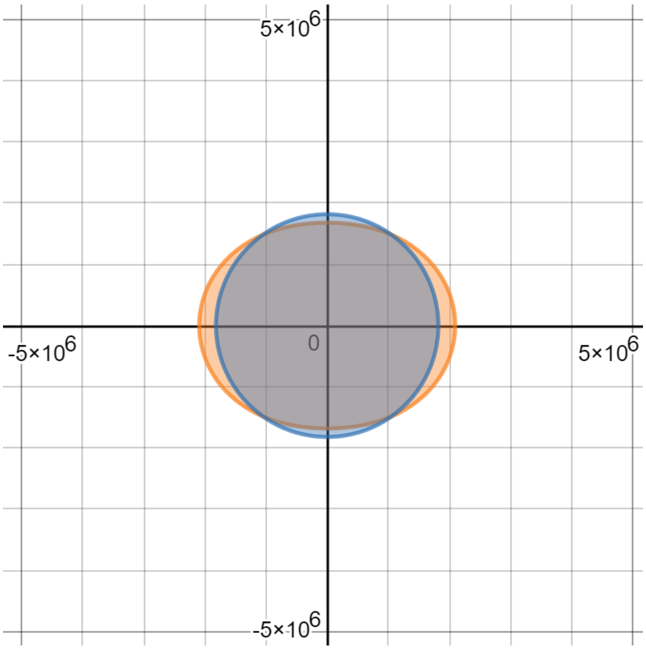

FIG. 4. A polar plot of the deformed Io (given by eq. (10)) overlaid with a spherical Io. The deformation is exaggerated.

B. Estimating Io's maximum tidal amplitude

Now that we know tidal warping as a function of angle (eq. (10)) we will find the maximum tidal amplitude of Io. The maximum tidal amplitude Δh_f is the difference in the height of Io's tidal bulge between perijove and apojove.

We're interested in the tidal bulge i.e. the tidal height at the point on Io closest to Jupiter, so $\alpha = 0 \implies \cos\alpha = 1$. Therefore eq. (10) gives

$$h_f = \frac{M_J R^4}{M_I L^3}.$$

The maximum tidal amplitude (assuming h_f at $\alpha = 0$ is greatest at perijove and least at apojove) is therefore

$$\Delta h_f = \frac{M_J R^4}{M_I L_P^3} - \frac{M_J R^4}{M_I L_A^3} = \frac{M_J R^4}{M_I}\left(\frac{1}{L_P^3} - \frac{1}{L_A^3}\right)$$

where L_P is the Io-Jupiter separation at perijove and L_A is the Io-Jupiter separation at apojove. This leads to a value of

$$\Delta h_f = 75.54\,\mathrm{m}.$$

Note that this value is calculated assuming Io is a fluid, and so is likely an overestimate. The observed height of Io's tidal bulge is about 50 m [4] and so our prediction is quite accurate.

III. QUALITATIVE MODEL OF THE INTERIOR OF IO

There is very little that we know for certain about the interior of Io. The most useful measurements come from the flybys of various spacecraft such as Pioneer 10 and Voyager. They were able to accurately measure Io's mass and size leading to the first estimate of Io's density, now accepted to be $3.53 \times 10^3\,\mathrm{kg\,m^{-3}}$, which is the highest of any moon in the solar system. Later on, during Galileo's 1999 flyby, the onboard magnetometer recorded measurements of the magnetic field along its trajectory giving us valuable insight into what the core of Io consists of.

We begin with a very simple model like that of Peale *et al.* [7] in order to find an approximation for the size of Io's core. We might hypothesise that the large majority of Io's mass comes from an abundance of iron and silicate rock, which are both extremely common in most terrestrial objects such as the rocky planets and the Moon. In this case we can estimate the volume ratio of the two as follows.

Let ρ_I and ρ_S be the densities of iron and silicate rock respectively. If there is a volume V_I of iron and a volume V_S of silicate rock in Io, then

$$\frac{\rho_I V_I + \rho_S V_S}{V_{\mathrm{Io}}} = \rho_{\mathrm{Io}}$$

and

$$V_I + V_S = V_{\mathrm{Io}}$$

where ρ_{Io} and V_{Io} are the density and volume of Io respectively. Therefore,

$$\rho_I \times V_I + \rho_S(V_{\mathrm{Io}} - V_I) = \rho_{\mathrm{Io}} V_{\mathrm{Io}}$$
$$\implies V_I = \frac{\rho_{\mathrm{Io}} - \rho_S}{\rho_I - 1} V_{\mathrm{Io}}.$$

If we assume an almost entirely pure iron core of density $7.9 \times 10^3\,\mathrm{kg\,m^{-3}}$ and take the density of silicate rock as $3.0 \times 10^3\,\mathrm{kg\,m^{-3}}$, then the volume of iron is

$$V_I = 1.5 \times 10^{18}\,\mathrm{m}^3$$

which gives an iron core of radius 710 km. This value is within the accepted range of 600–900 km depending on the concentration of sulphur compounds in the core.

Now in order to improve our model, we look at the plausibility of Io having a subsurface ocean of free iron much like the model by Schubert *et al.* [8]. The first clue that there may lie a molten layer beneath the surface is the active volcanoes that most likely form above molten pockets of rock. However until the investigation done by Khurana *et al.* [9], there was a real lack of scientific evidence to support this. Khurana looked at the warping of Jupiter's rotating magnetic field and in turn noted that Io must contain a global conducting layer. A field decrease of nearly 40 percent of the background Jovian field at closest approach to Io was recorded by the Galileo spacecraft. Kivelson *et al.* [10] showed that plasma sources alone would not warp the field to such an extent but instead an amount of free iron must be present for the induced source. This would act as a conducting layer allowing an induced field to occur throughout Io and in turn have the affect of weakening Jupiter's field nearby. They went on to estimate the lower bound for the thickness of this layer to be around 50 km. The presence of such a fluid layer would also increase the accuracy of a fluid approximation to Io's deformation under tidal forces.

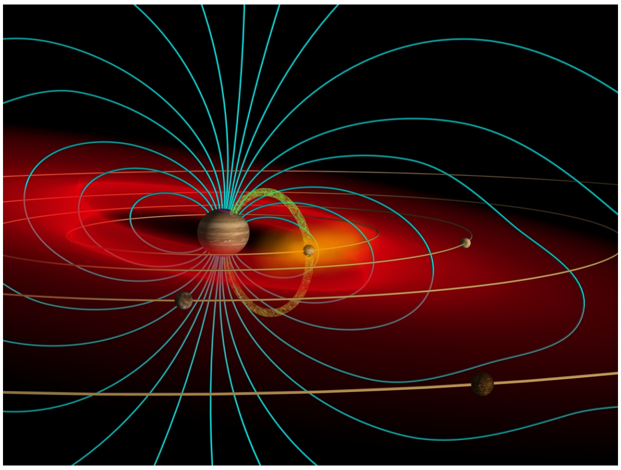

FIG. 5. The magnetosphere of Jupiter. Image from Khurana *et al.* [9].

There is still an element of ambiguity however in that a magnetised core could have the same effect as a global subsurface layer. Evidence both for and against this hypothesis is lacking however and in favour of the more studied model, we will assume the Schubert model.

Io's lithosphere is better understood on the other hand. It consists of a combination of silicate rocks and alkali-rich minerals such as feldspars and nepheline. At the base of this, we begin to see the melting of the rocks to form the magma.

As for the mantle, in the region of 700 - 1750 km from the centre, we know temperatures in the asthenosphere must exceed 1400 K in order to support rock melting whilst the rest of the mantle is solid and silicate rich.

In conclusion, we see that Io is in fact a lot more like terrestrial bodies than other moons in the size of its core and the presence of the molten asthenosphere.

IV. TIDAL DEFORMATION AS A FUNCTION OF TIME

There will be two main effects as Io completes each orbit. Firstly, the varying distance to Jupiter will cause the height of the tidal bulge to change. Secondly, because Io is tidally locked with Jupiter but its orbit is eccentric, the bulge will change position on Io's surface over time, also causing warping. We will ignore the latter effect for the purposes of this paper as its contribution to heating may be considered negligible.

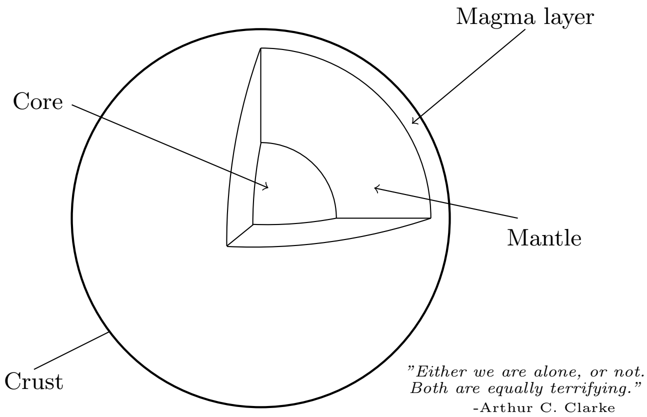

FIG. 6. A diagram showing the inferred structure of Io's interior.

A. Io-Jupiter separation as a function of time

The distance of a body in elliptical orbit to the centre of the body it is orbiting is

$$L(\theta) = a \cdot \frac{1 - e^2}{1 + e\cos\theta}, \tag{11}$$

where a is the semi-major axis, e the eccentricity of the orbit and θ is the true anomaly. The relation between the true anomaly and the eccentric anomaly E is

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E}.$$

Substituting this into eq. (11), we get

$$\begin{aligned} L(E) &= a \cdot \frac{(1-e^2)}{1 + e\dfrac{\cos E - e}{1 - e\cos E}} \\ &= a \cdot \frac{(1-e^2)(1-e\cos E)}{(1-e\cos E) + e(\cos E - e)} \\ &= a \cdot \frac{(1-e^2)(1-e\cos E)}{1-e^2} \\ \implies L(E) &= a(1 - e\cos E). \end{aligned} \tag{12}$$

Kepler's equation gives

$$M = E - e\sin E$$

where M is the mean anomaly, and so since Io's eccentricity e is small we may approximate $M \approx E$. The mean anomaly is given by

$$M = \omega t$$

if Io is at perijove when $t = 0$. Thus, eq. (12) gives the Io-Jupiter distance as

$$L(t) = a(1 - e\cos\omega t).$$

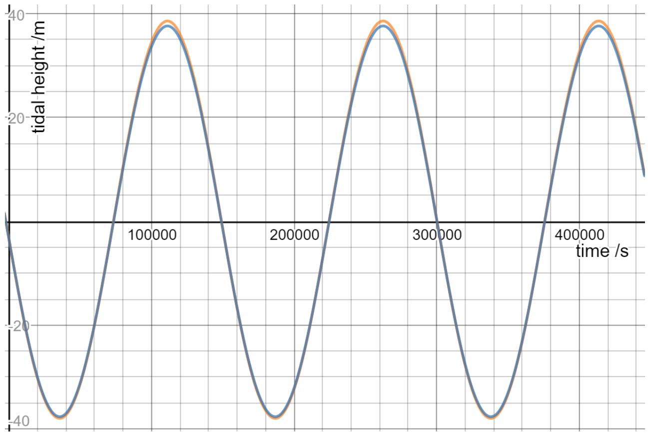

FIG. 7. A plot of fluid tidal height h_f as a function of time at $\alpha = 0$.

B. Tidal height as a function of time

Now combining this expression with eq. (10), the fluid tidal height of a point on Io's surface as a function of time is

$$h_f(t) = \frac{M_J}{M_I}\frac{R^4}{2L^3(t)}(3\cos^2\alpha - 1)$$
$$= \frac{M_J}{M_I}\frac{R^4(3\cos^2\alpha - 1)}{2a^3(1 - e\cos\omega t)^3}.$$

A plot of this function over time is shown in fig. 7.

C. Making simple harmonic approximations

From fig. 7 it is evident that Io's tidal warping behaves almost exactly as a sinusoidal wave. This shows us that we can approximate Io's behaviour as that of a simple harmonic oscillator, a class of problems very well-studied. We will make this approximation in the next section.

V. HEAT GENERATED BY TIDAL DEFORMATION

We have modelled Io's deformation as a function of time, and have shown that its behaviour is approximately simple harmonic. The power lost to heat as a result of tidal forces at any point in Io is simply the dot product of the velocity of that point and the resistive force at that point.

We will therefore model both the velocity and resistive forces as vector sinusoidal, plus a constant, as a function of both position within Io and of time. Where $\overrightarrow{F_F}$ is the resistive force per unit mass and $\vec{v}$ is the velocity, we will show that the power dissipated at any moment in time by the whole of Io is therefore approximately

$$P_T = \iiint_V \vec{v} \cdot \overrightarrow{F_F}\rho\,\mathrm{d}V$$

where ρ is the density of Io. We will then go on to temporally average this over a single time period, to find the average tidal power dissipated by Io.

A. Finding the vector simple harmonic form of velocity

For the purposes of this section we will define a new angle ε as shown in figs. 8 and 9 and will use the spherical coordinate system indexed by r, α and ε.

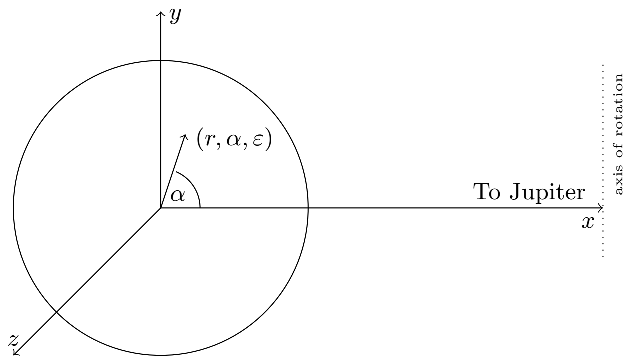

FIG. 8. A side view of our coordinate system.

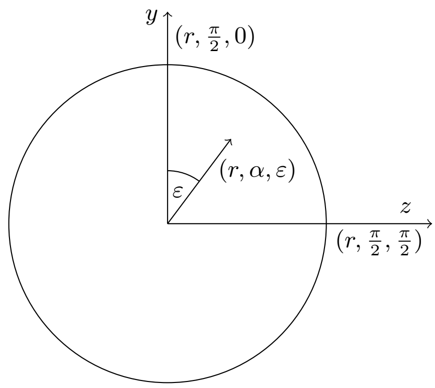

FIG. 9. A head-on view of our coordinate system.

We know from section IV that the tidal height h approximates a sinusoidal wave. The amplitude of this wave is just half the difference between tidal height at perijove and tidal height at apojove. Since eq. (10) gives the tidal height as

$$h = \frac{M_J}{M_I}\cdot\frac{R^4}{2L^3}\left(3\cos^2\alpha - 1\right),$$

the tidal amplitude is therefore

$$\implies h_0(\alpha,\varepsilon) = \frac{M_J}{M_I}\cdot\frac{R^4}{4}\left(3\cos^2\alpha - 1\right)\left(\frac{1}{L_\mathrm{P}^3} - \frac{1}{L_\mathrm{A}^3}\right)$$

where L_P and L_A are the distances between the centres of Io and Jupiter at perijove and apojove respectively. Therefore, defining $t = 0$ to be at apojove, the vector tidal height is

$$\vec{h}(\alpha,\varepsilon,t) = (-h_0(\alpha,\varepsilon)\cos(\omega t) + \overline{h}) \begin{pmatrix} \cos\alpha \\ \sin\alpha\cos\varepsilon \\ \sin\alpha\sin\varepsilon \end{pmatrix} = (-h_0(\alpha,\varepsilon)\cos(\omega t) + \overline{h})\hat{c} \tag{13}$$

where $\hat{c} := \begin{pmatrix} \cos\alpha \\ \sin\alpha\cos\varepsilon \\ \sin\alpha\sin\varepsilon \end{pmatrix}$ is the unit vector from any point (r,α,ε) radially outwards from the origin and $\overline{h}$ is the mean tidal height. Here our vectors take Io's centre as the origin, with the x axis through the centre of Jupiter, the y axis parallel to the axis of rotation and the z axis perpendicular to these two.

Now differentiating eq. (13), we find the velocity of a point on the surface of Io to be

$$\dot{\vec{h}}(\alpha,\varepsilon,t) = \omega h_0(\alpha,\varepsilon)\sin(\omega t)\hat{c}.$$

It is clear, however, that velocity will scale linearly with radius — that is, if a point (r,α,ε) has velocity $\dot{\vec{h}}$ then a point $(\frac{r}{2},\alpha,\varepsilon)$ will have velocity $\frac{\dot{\vec{h}}}{2}$ and so on. This principle is shown in fig. 10.

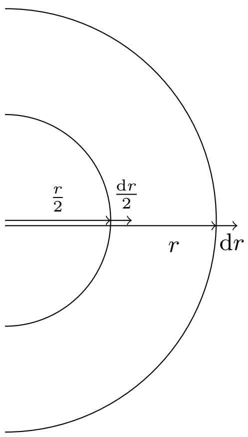

FIG. 10. If when a tide occurs the surface is extended by an amount $\mathrm{d}r$ then a point halfway out from the centre of Io will be extended by an amount $\frac{\mathrm{d}r}{2}$.

Therefore, the velocity at time t of any point inside Io (r,α,ε) is

$$\begin{aligned} \vec{v}(r,\alpha,\varepsilon,t) &= \frac{r}{R}\dot{\vec{h}}(\alpha,\varepsilon,t) \\ &= \frac{\omega r h_0(\alpha,\varepsilon)}{R}\cdot\sin(\omega t)\hat{c} \\ &= v_0\cdot\sin(\omega t)\hat{c} \end{aligned} \tag{14}$$

where $v_0 = \frac{\omega r h_0(\alpha,\varepsilon)}{R}$ is the magnitude of the velocity amplitude. Indeed, this will be maximised at $(r,\alpha,\varepsilon) = (R,0,\varepsilon)$ and so

$$v_{0_{\max}} \approx 0.001\,56\,\mathrm{m\,s^{-1}}$$

which appears to be a reasonable value.

B. Finding the vector simple harmonic form of force

Using our previous formula in eq. (7) for the potential energy V on the surface of Io and modifying it for any point (r,α,ε) in the interior of Io, we replace the M_I in the potential due to Io's gravitational field with $M_I\frac{r^3}{R^3}$ to account for the lessening of Io's gravitational potential in its interior due to less of its mass having an impact. This is a consequence of the shell theorem [11]. The potential energy becomes

$$\frac{V(r,\alpha,\varepsilon,L)}{m} = -\frac{1}{2}\omega^2 r^2 - \frac{GM_I\left(\frac{r}{R}\right)^3}{r} - \frac{GM_J\,r^2}{2L^3}\cdot(3\cos^2\alpha - 1) + C.$$

Since each concentric shell to the surface will be approximately equipotential, we know that the tidal force will be acting radially away from (or towards) Io's centre.

Therefore differentiating with respect to r, the total tidal force acting on a point mass m at (r,α,ε) will be

$$\begin{aligned} &\overrightarrow{F_T} = -\vec{\nabla}V \\ \implies &\frac{\overrightarrow{F_T}}{m} = \left(\omega^2 r + \frac{2GM_I\,r}{R^3} + \frac{GM_J\,r}{L^3}(3\cos^2\alpha - 1)\right)\hat{c}. \end{aligned}$$

Plotting a graph of tidal force against angle α shows that it can be approximated as simple harmonic in time, but off by a constant.

We will from now take all forces we talk about as being on a unit mass. The force can be expressed in harmonic form as

$$\overrightarrow{F_T} = (F_{T_{\mathrm{amp}}}\cos(\omega t) + \overline{F_T})\hat{c}$$

where the mean tidal force is

$$\overline{F_T} = \frac{GM_J r}{2\overline{L}^3}(3\cos^2\alpha - 1).$$

and the amplitude of the tidal force $F_{T_{\mathrm{amp}}}$ is

$$F_{T_{\mathrm{amp}}}(r,\alpha,\varepsilon) = \frac{GM_J r}{2}(3\cos^2\alpha - 1)\left(\frac{1}{L_P^3} - \frac{1}{L_A^3}\right). \tag{15}$$

The displacement of any point in Io is approximately sinusoidal in time. This means that the acceleration at any point will likewise be sinusoidal. Since for any point mass at arbitrary position acceleration is proportional to the force applied by Newton's second law of motion (see Newton [11]), we know that the resultant force must be likewise sinusoidal. Therefore, in some time convention t, the resultant force on a unit mass is

$$\overrightarrow{F_R} = \left(\frac{\omega^2 r h_0(\alpha,\varepsilon)}{R}\cdot\cos(\omega t)\right)\hat{c}.$$

Since the only forces acting on any point (r, α, ε) are the tidal forces and the material frictional tension forces, this allows us to calculate the frictional force, as the motion of the object and the tidal force are both sinusoidal with some phase difference ζ. The frictional force is therefore

$$\begin{aligned}
\overrightarrow{F_F} &= \overrightarrow{F_R} - \overrightarrow{F_T} \\
\implies \overrightarrow{F_F} m &= \left(\frac{\omega^2 r h_0(\alpha, \varepsilon)}{R} \cdot \cos(\omega t)\right)\hat{c} \\
&\quad - \left(F_{T_{amp}} \cos(\omega t + \zeta) + \overline{F_T}\right)\hat{c} \\
&= \left(F_0 \cos(\omega t + \delta) - \overline{F_T}\right)\hat{c}, \qquad (16)
\end{aligned}$$

where

$$F_0 = \sqrt{\omega^2 v_0^2 + F_{T_{\mathrm{amp}}}^2} \qquad (17)$$

is the amplitude of oscillation of the frictional force, and $\delta = \frac{\pi}{2} + \zeta$ is the phase difference between the velocity and force oscillations.

C. Calculating power dissipated as a function of phase difference

We now have harmonic expressions for force (on a mass m) and velocity in eqs. (14) and (16). To calculate the average power dissipated by tidal forces, we have to integrate over all points (r, α, ε) in Io and then take the temporal average over one time period T:

$$\langle P_T \rangle = \frac{1}{T}\int_0^T \iiint_V \vec{v} \cdot \overrightarrow{F_F} \rho \, \mathrm{d}V \, \mathrm{d}t.$$

Here $\mathrm{d}V$ is a small volume of Io, so that where ρ is the density of Io, $\rho \, \mathrm{d}V$ is the small mass on which the force acts. The volume element in our spherical polar coordinate system is $\mathrm{d}V = r^2 \sin\alpha \, \mathrm{d}r \, \mathrm{d}\varepsilon \, \mathrm{d}\alpha$ and so this integral becomes

$$\langle P_T \rangle = \frac{1}{T}\int_0^T \int_0^\pi \int_0^{2\pi} \int_0^R \vec{v} \cdot \overrightarrow{F_F} \rho r^2 \sin\alpha \, \mathrm{d}r \, \mathrm{d}\varepsilon \, \mathrm{d}\alpha \, \mathrm{d}t.$$

We now substitute in our expressions for frictional force per unit mass $\overrightarrow{F_F}$ and velocity $\vec{v}$ from eqs. (14) and (16) and rearrange the integral:

$$\begin{aligned}
\langle P_T \rangle &= \frac{1}{T}\int_0^T \int_0^\pi \int_0^{2\pi} \int_0^R v_0 \cos(\omega t) \left[F_0 \cos(\omega t + \delta) - \overline{F_T}\right] \rho r^2 \sin\alpha \, \mathrm{d}r \, \mathrm{d}\varepsilon \, \mathrm{d}\alpha \, \mathrm{d}t \\
&= \frac{1}{T}\int_0^T \cos(\omega t)\cos(\omega t + \delta) \int_0^\pi \int_0^{2\pi} \int_0^R \rho r^2 \sin\alpha v_0 F_0 \, \mathrm{d}r \, \mathrm{d}\varepsilon \, \mathrm{d}\alpha \, \mathrm{d}t \\
&\quad - \frac{1}{T}\int_0^T \cos(\omega t) \int_0^\pi \int_0^{2\pi} \int_0^R v_0 \overline{F_T} \, \mathrm{d}r \, \mathrm{d}\varepsilon \, \mathrm{d}\alpha \, \mathrm{d}t \\
&= \frac{1}{T}\int_0^T \cos(\omega t)\cos(\omega t + \delta) \, \mathrm{d}t \int_0^\pi \int_0^{2\pi} \int_0^R \rho r^2 \sin\alpha v_0 F_0 \, \mathrm{d}r \, \mathrm{d}\varepsilon \, \mathrm{d}\alpha \, . \qquad (18)
\end{aligned}$$

The second half of the penultimate line above is discarded because the temporal average of $\cos(\omega t)$ over one time period is zero. Looking at eq. (18), we now define the temporal part as

$$I_t := \frac{1}{T}\int_0^T \cos(\omega t)\cos(\omega t + \delta) \, \mathrm{d}t$$

and the spatial part as

$$I_s := \int_0^\pi \int_0^{2\pi} \int_0^R \rho r^2 \sin\alpha v_0 F_0 \, \mathrm{d}r \, \mathrm{d}\varepsilon \, \mathrm{d}\alpha$$

so that the average power is

$$\langle P_T \rangle = I_t I_s.$$

We start by evaluating I_t:

$$\begin{aligned}
I_t &= \frac{1}{T}\int_0^T \cos(\omega t)\cos(\omega t + \delta) \, \mathrm{d}t \\
&= \frac{1}{T}\int_0^T \cos(\omega t)(\cos(\omega t)\cos\delta - \sin(\omega t)\sin\delta) \, \mathrm{d}t \\
&= \frac{1}{T}\int_0^T \cos^2(\omega t)\cos\delta - \frac{1}{2}\sin(2\omega t)\sin\delta \, \mathrm{d}t \\
&= \frac{1}{T}\left[\frac{\cos\delta}{2\omega}(\omega t + \sin(\omega t)\cos(\omega t)) + \frac{\sin\delta}{4\omega}\cos(2\omega t)\right]_0^T \\
&= \frac{1}{T}\left(\frac{T\cos\delta}{2} + \frac{\sin\delta}{4\omega} - \frac{\sin\delta}{4\omega}\right) \\
&= \frac{\cos\delta}{2}.
\end{aligned}$$

Now we will evaluate the spatial part I_s. We previously derived that the mean velocity is

$$v_0 = \frac{\omega r h_0}{R} = \frac{\omega r}{R}\frac{M_J}{M_I}\cdot\frac{R^4}{4}\left(3\cos^2\alpha - 1\right)\left(\frac{1}{L_{\mathrm{P}}^3} - \frac{1}{L_{\mathrm{A}}^3}\right)$$

and so we define

$$k_v := \frac{\omega}{R}\frac{M_J}{M_I}\cdot\frac{R^4}{4}\left(\frac{1}{L_{\mathrm{P}}^3} - \frac{1}{L_{\mathrm{A}}^3}\right)$$

such that

$$v_0 = k_v r(3\cos\alpha - 1)^2.$$

Similarly, the force amplitude is given by eq. (15) as

$$F_{T_{\mathrm{amp}}} = \frac{GM_J r}{2}(3\cos^2\alpha - 1)\left(\frac{1}{L_P^3} - \frac{1}{L_A^3}\right)$$

so we define

$$k_F := \frac{GM_J}{2}\left(\frac{1}{L_P^3} - \frac{1}{L_A^3}\right)$$

such that

$$\implies F_{T_{\mathrm{amp}}} = k_F r(3\cos^2\alpha - 1).$$

Therefore, our frictional force amplitude F_0 is given by eq. (17) as

$$\begin{aligned} F_0 &= \sqrt{\omega^2 v_0^2 + F_{T_{\mathrm{amp}}}^2} \\ &= \sqrt{\omega^2 k_v^2 r^2 (3\cos^2\alpha - 1)^2 + k_F^2 r^2 (3\cos^2\alpha - 1)^2} \\ &= r(3\cos^2\alpha - 1)\sqrt{\omega^2 k_v^2 + k_F^2}. \end{aligned}$$

This means that our spatial integral takes the form

$$\begin{aligned} I_s &= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho r^2 \sin\alpha v_0 F_0 \,\mathrm{d}r\ \mathrm{d}\varepsilon\ \mathrm{d}\alpha \\ &= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho r^2 \sin\alpha k_v r(3\cos\alpha - 1)^2 r(3\cos^2\alpha - 1)\sqrt{\omega^2 k_v^2 + k_F^2}\,\mathrm{d}r\ \mathrm{d}\varepsilon\ \mathrm{d}\alpha \\ &= 2\pi\int_0^{\pi}\int_0^{R} r^4 \sin\alpha (3\cos^2\alpha - 1)^2 \left(\rho k_v\sqrt{\omega^2 k_v^2 + k_F^2}\right)\mathrm{d}r\ \mathrm{d}\alpha. \end{aligned}$$

Defining

$$k_I := \rho k_v \sqrt{\omega^2 k_v^2 + k_F^2},$$

this integral becomes

$$\begin{aligned} I_s &= 2\pi\int_0^{\pi}\int_0^{R} r^4 \sin\alpha(3\cos^2\alpha - 1)^2 k_I \,\mathrm{d}r\ \mathrm{d}\alpha \\ &= 2\pi k_I \frac{R^5}{5}\int_0^{\pi} \sin\alpha(3\cos^2\alpha - 1)^2\,\mathrm{d}\alpha \\ &= 2\pi k_I \frac{R^5}{5}\left[-\frac{9\cos^5\alpha}{5} + 2\cos^3\alpha - \cos\alpha\right]_0^{\pi} \\ &= 2\pi k_I \frac{8R^5}{25}. \end{aligned}$$

This leaves us the following formula for the power loss in terms of the phase difference:

$$\begin{aligned}
\langle P_T \rangle &= \pi k_I \frac{8R^5}{25} \cos\delta \\
&= \pi \frac{8R^5}{25} \cos\delta \cdot \rho \frac{M_J}{M_I} \cdot \frac{\omega R^3}{4} \left(\frac{1}{L_{\mathrm{P}}^3} - \frac{1}{L_{\mathrm{A}}^3} \right) \sqrt{w^2 \left(\frac{M_J}{M_I} \cdot \frac{\omega R^3}{4} \left(\frac{1}{L_{\mathrm{P}}^3} - \frac{1}{L_{\mathrm{A}}^3} \right) \right)^2 + \left(\frac{GM_J}{2} \left(\frac{1}{L_P^3} - \frac{1}{L_A^3} \right) \right)^2} \\
&= \rho \frac{\pi \omega R^8}{25} \frac{M_J^2}{M_I} \left(\frac{1}{L_P^3} - \frac{1}{L_A^3} \right)^2 \sqrt{\frac{\omega^4 R^6}{4M_I^2} + G^2} \cdot \cos\delta \\
&= \rho \frac{\pi \omega R^8}{25} \frac{M_J^2}{M_I^2} \left(\frac{1}{L_P^3} - \frac{1}{L_A^3} \right)^2 \sqrt{\frac{\omega^4 R^6}{4} + (GM_I)^2} \cdot \cos\delta \\
&= \rho \frac{\pi \omega R^9}{25} \frac{M_J^2}{M_I^2} \left(\frac{1}{L_P^3} - \frac{1}{L_A^3} \right)^2 \sqrt{\left(\frac{\omega^2 R^2}{2} \right)^2 + \left(\frac{GM_I}{R} \right)^2} \cdot \cos\delta.
\end{aligned} \tag{19}$$

Therefore, the maximum value that could possibly be obtained for the tidal thermal power would be

$$\begin{aligned}
\langle P_T \rangle_{max} &= \pi k_I \frac{8R^5}{25} \\
&\approx 6.1876 \cdot 10^{14} \mathrm{W}.
\end{aligned}$$

If Io were a perfectly elastic body, the resistive force would be exactly $\frac{\pi}{2}$ out of phase since the acceleration and tidal forces would be exactly in phase. This would give us, by our formula, a power loss of zero, which fits with our conventional understanding of perfectly elastic materials.

D. Estimating average power dissipated

The phase angle δ between the tidal force and the velocity is dependent on the viscoelasticity of the interior of Io. For perfectly elastic oscillators, the tidal force and the resulting deformation will in phase (so $\delta = \frac{\pi}{2}$), and for perfectly viscous oscillators, the tidal force and the resulting deformation will be $\frac{\pi}{2}$ out of phase (so $\delta = \pi$).

We can realistically expect the actual phase difference to be somewhere in between. Phase difference between force and response is inversely proportional to the tidal quality factor Q [12] which is a notoriously difficult value to calculate. The tidal quality factor is generally smaller for smaller bodies, and so we will make an approximate order-of-magnitude estimate of $Q \approx 10$. This leads to a phase lag of $\delta = \frac{\pi}{2} - \frac{1}{10} = 84.2°$. The resulting power dissipated is given by eq. (19) as

$$\begin{aligned}
\langle P_T \rangle &= \pi k_I \frac{8R^5}{25} \cos(84.2°) \\
&= 6.3 \times 10^{13}\,\mathrm{W}.
\end{aligned} \tag{20}$$

Dividing this by the surface area of Io leads to a predicted surface heat flux of $1.5\,\mathrm{W\,m^{-2}}$ which is in good agreement with the observed value of 1 to $2\,\mathrm{W\,m^{-2}}$ [1].

E. Surface temperature of Io

The power absorbed by Io from the Sun is

$$P_{\mathrm{Sun}} = \frac{L_{\mathrm{Sun}} R^2 (1-A)}{4D^2} = 1.94 \times 10^{14}\,\mathrm{W}$$

where $L_{\mathrm{Sun}} = 3.83 \times 10^{26}\,\mathrm{W}$ is the luminosity of the Sun and $A = 0.63$ is Io's albedo. Combining this with tidal heating in eq. (20), the total power absorbed by Io is

$$P_{\mathrm{in}} = 2.57 \times 10^{14}\,\mathrm{W}.$$

It is worth noting that this total power input is if anything an underestimate as we have neither taken into account the tidal heating caused by the movement of Io's tidal bulge across its surface (cf. section IV) nor the effects of other heat sources such as radiogenic heating.

Io radiates as a blackbody and so the Stefan-Boltzmann law gives the power radiated as

$$P_{\mathrm{out}} = 4\pi R^2 \sigma T^4$$

where σ is the Stefan-Boltzmann constant and T is the effective temperature of Io. Setting $P_{\mathrm{in}} = P_{\mathrm{out}}$ yields an effective surface temperature of

$$T = \sqrt[4]{\frac{2.57 \times 10^{14}\,\mathrm{W}}{4\pi R^2 \sigma}} = 103\,\mathrm{K}.$$

This is in good agreement with the observed mean surface temperature of 110 K. An estimate not taking tidal heating into account yields a value of 96.1 K.

Therefore tidal heating does not actually cause a significant rise in the effective temperature of Io, but because of the large amount of internally generated heat there is extreme volcanic activity on the moon.

VI. ORBITAL RESONANCE OF THE GALILEAN SATELLITES

We know that Io loses about $6.3 \times 10^{13}\,\mathrm{W} \cdot (365.25 \cdot 24 \cdot 3600\,\mathrm{s}) = 2.0 \times 10^{21}\,\mathrm{J}$ to heat through the tides every Earth year. While tidal heating would usually result in the decay of the moon's angular momentum and orbital decay as energy from a moon's elliptic orbit and spin is converted into tidal heating, Io's orbit is both already synchronous and continuously eccentric.

By the principle of conservation of energy, we therefore know that this much energy must be put into the Ionian system every year by an external agent. Given that Io is in a synchronous orbit, the energy input must come from the eccentricity of Io's orbit, and therefore the forces that keep Io's orbit elliptical are the sources of the tidal heating of Io. We also know that Io's orbital period is exactly twice that of Europa and exactly four times that of Ganymede. Io, as a result, experiences an oscillating force from these two moons, which keeps Io's orbit eccentric.

It is therefore likely that the orbits of Io, Europa and Ganymede will circularise over time as the energy from their elliptic orbit is transferred to Io through gravitation and then dissipated through tidal heating.

VII. CONCLUSION

In this paper, we have used classical methods to find the deformation of Io under Jupiter. Using only simple experimental measurements, we found an estimate of the surface temperature of Io due to tidal heating.

This is particularly significant, as these are measurements that we can take for any moon system with a stable elliptic orbit. Therefore, this method can be extended for any such system to find theoretical surface temperature if tidal heating were the main heating effect (apart from the system's central star). If the recorded mean temperature were significantly lower than that theorised by tidal heating, we could conclude that the material of the planet was significantly stiffer or softer than that which would optimise heat generated. On the other hand, if the recorded mean temperature were significantly higher, we would know that other factors such as radiogenic heating are dissipating heat energy, and so we could know to study the body in more detail.

Therefore, the modelling techniques detailed in this paper are not only accurate in the context of Ionian tidal heating, but could be extended in studying moons in extrasolar systems to determine the material properties of those moons. The natural continuation of this technique would be to find some explicit formula for the all-important phase difference δ in terms of better known properties of materials and structures, such as Young's modulus and shear modulus. Given such a link, we would be able to analyse with very few measurements the materials with which any moon was made.

This paper is clearly a step towards understanding not only the qualitative reasoning behind tidal analysis, but also creating a solid, quantitative approach that can allow others to more accurately study the properties of planetary bodies.

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