AN INVESTIGATION INTO ELECTRIC FIELDS AROUND CHARGED SPHERES

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I A HOLLOW CHARGED SPHERE

Consider a thin, spherical shell of radius r and surface charge density σ. To find the charge at a point h from its centre, we note that by symmetry all tangential fields cancel and so we need only consider the radial component of the electric field at this point.

Suppose the shell is centred at the origin in $\mathbb{R}^3$ and that our test point is on the z-axis at position

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

so that any point on the shell has position vector

$$\mathbf{r} = \begin{pmatrix} r\cos\phi\cos\theta \\ r\cos\phi\sin\theta \\ r\sin\phi \end{pmatrix}$$

where θ and ϕ are the two angles shown in fig. 1.

By Coulomb's law, the contribution of a small amount of charge at point $\mathbf{r}$ to the field at $\mathbf{h}$ is

$$\mathrm{d}E = \frac{\mathrm{d}Q}{4\pi\varepsilon_0|\mathbf{h}-\mathbf{r}|^2}.$$

By definition of surface charge density,

$$\sigma = \frac{\mathrm{d}Q}{\mathrm{d}A}$$

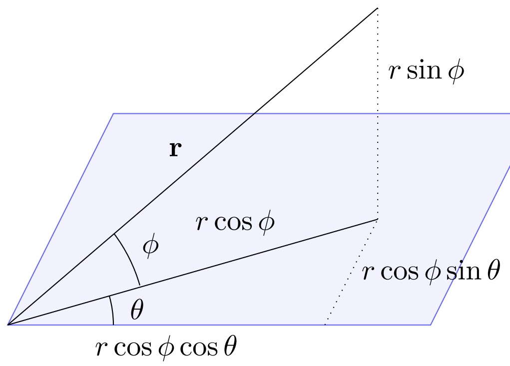


Figure 1: A point r on a thin spherical shell.

where $\mathrm{d}A$ is a small area of the shell at $\mathbf{r}$, and it can be seen that this small area has 'height' $r \cdot \mathrm{d}\phi$ and 'width' $r \cos \phi \cdot \mathrm{d}\theta$ (both arc lengths), so that

$$\mathrm{d}E = \frac{\sigma r^2 \cos \phi \, \mathrm{d}\theta \, \mathrm{d}\phi}{4\pi\varepsilon_0 |\mathbf{h} - \mathbf{r}|^2}.$$

As explained, we only want the radial (vertical) component of this field, and so we multiply by $\cos \psi$ where ψ is the angle $\mathbf{h} - \mathbf{r}$ makes to the vertical:

$$\mathrm{d}E_z = \frac{\sigma r^2 \cos \phi \, \mathrm{d}\theta \, \mathrm{d}\phi}{4\pi\varepsilon_0 |\mathbf{h} - \mathbf{r}|^2} \cos \psi.$$

Therefore by the principle of superposition, the total field at point $\mathbf{h}$ is

$$E = E_z = \frac{\sigma r^2}{4\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \frac{\cos \phi \cos \psi}{|\mathbf{h} - \mathbf{r}|^2} \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

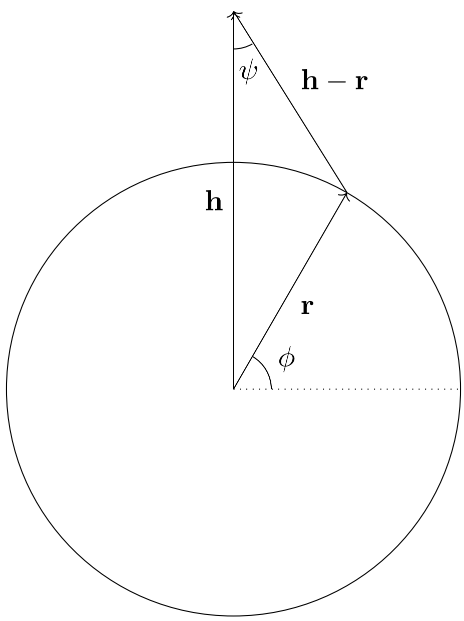


Figure 2: A vertical slice through the sphere (the situation is of course rotationally symmetric around the z-axis).

From fig. 2 it can be seen that

$$\cos \psi = \frac{\mathbf{h}_z - \mathbf{r}_z}{|\mathbf{h} - \mathbf{r}|} = \frac{h - r \sin \phi}{|\mathbf{h} - \mathbf{r}|}$$

and so our value for the total electric field becomes

$$E = \frac{\sigma r^2}{4\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \frac{\cos\phi(h - r\sin\phi)}{|\mathbf{h} - \mathbf{r}|^3} \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

The situation is rotationally symmetric around the z-axis, which means nothing can be a function of θ. Therefore performing the inner integral is a trivial matter of multiplying by 2π:

$$E = \frac{\sigma r^2}{2\varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\phi(h - r\sin\phi)}{|\mathbf{h} - \mathbf{r}|^3} \, \mathrm{d}\phi.$$

Next, we want to find the denominator:

$$\begin{aligned}
|\mathbf{h} - \mathbf{r}| &= \left| \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \begin{pmatrix} r\cos\phi\cos\theta \\ r\cos\phi\sin\theta \\ r\sin\phi \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} -r\cos\phi\cos\theta \\ -r\cos\phi\sin\theta \\ h - r\sin\phi \end{pmatrix} \right| \\
&= \sqrt{r^2\cos^2\phi\cos^2\theta + r^2\cos^2\phi\sin^2\theta + h^2 - 2hr\sin\phi + r^2\sin^2\phi} \\
&= \sqrt{r^2\cos^2\phi + h^2 - 2hr\sin\phi + r^2\sin^2\phi} \\
&= \sqrt{r^2 + h^2 - 2hr\sin\phi}
\end{aligned}$$

which, as expected, is not a function of θ. So, the integral we need to evaluate is

$$E = \frac{\sigma r^2}{2\varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\phi(h - r\sin\phi)}{\left(h^2 + r^2 - 2hr\sin\phi\right)^{3/2}} \, \mathrm{d}\phi.$$

Substituting $u = \sin\phi$ so that $\mathrm{d}u = \cos\phi \, \mathrm{d}\phi$, this becomes

$$\begin{aligned}
E &= \frac{\sigma r^2}{2\varepsilon_0} \int_{-1}^{1} \frac{h - ru}{\left(h^2 + r^2 - 2hru\right)^{3/2}} \, \mathrm{d}u \\
&= \frac{\sigma r^2 h}{2\varepsilon_0} \int_{-1}^{1} \frac{1}{\left(h^2 + r^2 - 2hru\right)^{3/2}} \, \mathrm{d}u - \frac{\sigma r^3}{2\varepsilon_0} \int_{-1}^{1} \frac{u}{\left(h^2 + r^2 - 2hru\right)^{3/2}} \, \mathrm{d}u. \qquad (1)
\end{aligned}$$

The first integral evaluates to

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\left(h^2 + r^2 - 2hru\right)^{3/2}} \, \mathrm{d}u &= \left[\frac{-2}{-2hr\sqrt{h^2 + r^2 - 2hru}} \right]_{-1}^{1} \\
&= \frac{1}{hr} \left(\frac{1}{\sqrt{h^2 + r^2 - 2hr}} - \frac{1}{\sqrt{h^2 + r^2 + 2hr}} \right) \\
&= \frac{1}{hr} \left(\frac{1}{|h - r|} - \frac{1}{h + r} \right). \qquad (2)
\end{aligned}$$

For the second integral we'll use parts, relying on the result we just showed:

$$\begin{aligned}
\int_{-1}^{1} \frac{u}{\left(h^2+r^2-2hru\right)^{3/2}}\,\mathrm{d}u &= \left[\frac{u}{hr\sqrt{h^2+r^2-2hru}}\right]_{-1}^{1} - \int_{-1}^{1} \frac{1}{hr\sqrt{h^2+r^2-2hru}}\,\mathrm{d}u \\
&= \left[\frac{u}{hr\sqrt{h^2+r^2-2hru}}\right]_{-1}^{1} - \frac{1}{hr}\left[\frac{2}{-2hr}\sqrt{h^2+r^2-2hru}\right]_{-1}^{1} \\
&= \frac{1}{hr}\left[\frac{u}{\sqrt{h^2+r^2-2hru}} + \frac{\sqrt{h^2+r^2-2hru}}{hr}\right]_{-1}^{1} \\
&= \frac{1}{hr}\left(\frac{1}{|h-r|} + \frac{1}{h+r} + \frac{|h-r|-(h+r)}{hr}\right) \qquad (3)
\end{aligned}$$

Putting the expressions from eqs. (2) and (3) together into our main expression for E in eq. (1), we come to

$$\begin{aligned}
E &= \frac{\sigma r}{2\varepsilon_0}\left(\frac{1}{|h-r|} - \frac{1}{h+r}\right) - \frac{\sigma r^2}{2\varepsilon_0 h}\left(\frac{1}{|h-r|} + \frac{1}{h+r} + \frac{|h-r|-(h+r)}{hr}\right) \\
&= \frac{\sigma}{2\varepsilon_0}\left(\frac{r}{|h-r|} - \frac{r}{h+r} - \frac{r^2}{h|h-r|} - \frac{r^2}{h(h+r)} - \frac{r|h-r|}{h^2} + \frac{r(h+r)}{h^2}\right) \\
&= \frac{\sigma}{2\varepsilon_0}\left(\frac{rh^2(h+r) - rh^2|h-r| - r^2h(h+r) - r^2h|h-r| - r|h-r||h-r|\,(h+r) + r(h+r)(h+r)|h-r|}{h^2(h+r)|h-r|}\right) \\
&= \frac{\sigma r}{2\varepsilon_0 h^2}\left(\frac{h^3 + rh^2 - h^2|h-r| - rh^2 - r^2h - rh|h-r| - |h-r|\left|h^2-r^2\right| + (h+r)\left|h^2-r^2\right|}{\left|h^2-r^2\right|}\right) \qquad (4)
\end{aligned}$$

From this rather complicated-looking expression there arise two simple cases: that where $h \leqslant r$ and that where $h > r$. If $h \leqslant r$, meaning our test point is *inside* the spherical shell, then

$$|h-r| = r-h$$

and

$$\left|h^2-r^2\right| = r^2-h^2.$$

Then, eq. (4) simplifies to

$$\begin{aligned}
E &= \frac{\sigma r}{2\varepsilon_0 h^2(r^2-h^2)}\left(h^3 + rh^2 - h^2(r-h) - rh^2 - r^2h - rh(r-h) - (r-h)(r^2-h^2) + (h+r)(r^2-h^2)\right) \\
&= \frac{\sigma r}{2\varepsilon_0 h^2(r^2-h^2)} \cdot 0 \\
&= 0.
\end{aligned}$$

This is wonderful: at no point within the sphere is there any electric field whatsoever! The second case, where $h > r$, means our test point is outside the sphere, and implies

$$|h-r| = h-r$$

and

$$\left|h^2-r^2\right| = h^2-r^2.$$

Therefore, eq. (4) reduces to:

$$\begin{aligned}
E &= \frac{\sigma r}{2\varepsilon_0 h^2(h^2-r^2)}\left(h^3 + rh^2 - h^2(h-r) - rh^2 - r^2h - rh(h-r) - (h-r)(h^2-r^2) + (h+r)(h^2-r^2)\right) \\
&= \frac{\sigma r}{2\varepsilon_0 h^2(h^2-r^2)}\left(2rh^2 - 2r^3\right) \\
&= \frac{\sigma r^2}{\varepsilon_0 h^2}. \qquad (5)
\end{aligned}$$

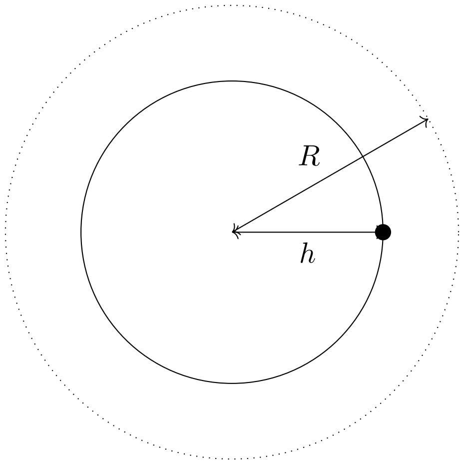


Figure 3: A smaller sphere of radius h within a larger one of radius R.

This is a beautiful result: the electric field depends, as makes sense intuitively, only on the ratio of the distance h to the radius r. It doesn't end there, though: note that the surface area of the sphere is $4\pi r^2$ and so where Q is the total charge,

$$\sigma = \frac{Q}{4\pi r^2}.$$

Then, eq. (5) becomes

$$E = \frac{Q}{4\pi\varepsilon_0 h^2};$$

the spherical shell generates the same electric field as would a point charge at its centre.

II A SOLID CHARGED SPHERE

Now that we know the behaviour of a hollow sphere, the case of a solid sphere is trivial. By splitting the solid sphere up into infinitely many shells, each with its equivalent charge at the centre of the sphere, it follows that the equivalent charge of the entire sphere must be at its centre and must have the value Q where Q is the total charge on the sphere. So, outside the sphere, the same formula applies:

$$E = \frac{Q}{4\pi\varepsilon_0 h^2}$$

at a distance h from the centre of the sphere.

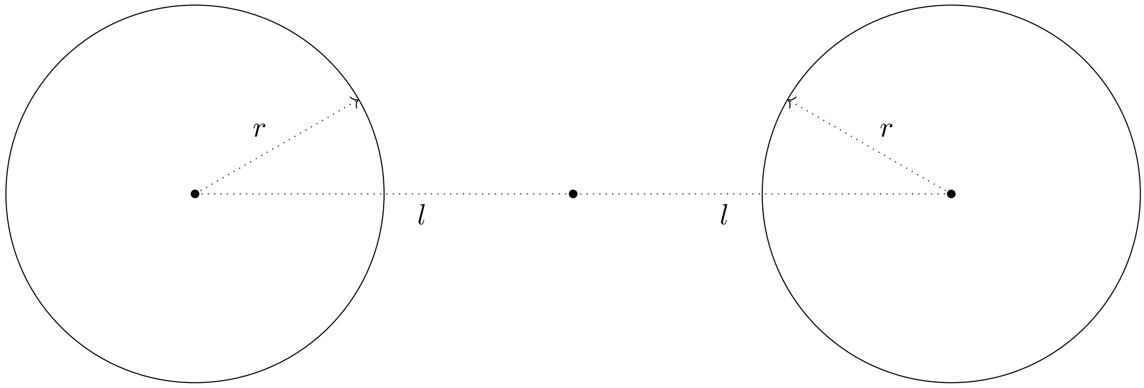
If the test charge is *within* the sphere, then as proven above, no subshell of radius r such that $h < r$ will exert a force on the test charge; we can disregard these shells so that we have a smaller solid sphere of radius h behaving in the same way.

If the original sphere has radius R then the new sphere is a fraction $\frac{h^3}{R^3}$ of the original volume, and so assuming the charge is uniformly distributed, the smaller sphere has a total charge of $\frac{h^3}{R^3}Q$. Consequently the electric field within the sphere is

$$E = \frac{\frac{h^3}{R^3}Q}{4\pi\varepsilon_0 h^2} = \frac{hQ}{4\pi\varepsilon_0 R^3}.$$

III THE ELECTRIC FIELD AROUND TWO SOLID CHARGED SPHERES

Now consider two insulating spheres, each with radius r and volume charge density ρ, with their centres separated by distance $2l$ such that $l > r$. We'll place the spheres along the z-axis in $\mathbb{R}^3$ so that the origin is midway between their centres.



At an arbitrary point $\mathbf{s} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we wish to determine the electric field $\mathbf{E}$.

There are three cases to consider: outside both spheres, inside the upper sphere, and inside the lower sphere.

Outside both spheres, each sphere can be modelled as a point charge at the centre of the sphere. The total charge of each sphere is $\rho V = \frac{4\pi r^3 \rho}{3}$ and these point charges are located at $\mathbf{z_1} = \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix}$ and $\mathbf{z_2} = \begin{pmatrix} 0 \\ 0 \\ -l \end{pmatrix}$. Hence, by the principle of superposition, the total field at a point $\mathbf{s}$ outside both spheres is

$$\mathbf{E} = \frac{\frac{4\pi r^3 \rho}{3}}{4\pi\varepsilon_0 |\mathbf{s} - \mathbf{z_1}|^3}(\mathbf{s} - \mathbf{z_1}) + \frac{\frac{4\pi r^3 \rho}{3}}{4\pi\varepsilon_0 |\mathbf{s} - \mathbf{z_2}|^3}(\mathbf{s} - \mathbf{z_2}) \tag{6}$$

$$= \frac{r^3\rho}{3\varepsilon_0 \left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix} \right|^3} \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix} \right] + \frac{r^3\rho}{3\varepsilon_0 \left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -l \end{pmatrix} \right|^3} \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -l \end{pmatrix} \right]$$

$$= \frac{r^3\rho}{3\varepsilon_0} \left[\frac{1}{\sqrt{x^2 + y^2 + (z-l)^2}^3} \begin{pmatrix} x \\ y \\ z - l \end{pmatrix} + \frac{1}{\sqrt{x^2 + y^2 + (z+l)^2}^3} \begin{pmatrix} x \\ y \\ z + l \end{pmatrix} \right]$$

$$= \frac{r^3\rho}{3\varepsilon_0} \left[\frac{1}{(x^2 + y^2 + z^2 - 2zl + l^2)^{3/2}} \begin{pmatrix} x \\ y \\ z - l \end{pmatrix} + \frac{1}{(x^2 + y^2 + z^2 + 2zl + l^2)^{3/2}} \begin{pmatrix} x \\ y \\ z + l \end{pmatrix} \right]. \tag{7}$$

When inside the upper sphere, the contribution from the lower sphere remains unchanged, but the electric field of the upper sphere decreases in magnitude as described in section II. The total equivalent charge of the upper sphere changes from $\frac{4}{3}\pi r^3 \rho$ to $\frac{4}{3}\pi |\mathbf{s}-\mathbf{z_1}|^3 \rho$, so the new electric field function (based on eq. (6)) is

$$\begin{aligned}\mathbf{E} &= \frac{\frac{4}{3}\pi|\mathbf{s}-\mathbf{z_1}|^3\rho}{4\pi\varepsilon_0|\mathbf{s}-\mathbf{z_1}|^3}(\mathbf{s}-\mathbf{z_1}) + \frac{\frac{4\pi r^3\rho}{3}}{4\pi\varepsilon_0|\mathbf{s}-\mathbf{z_2}|^3}(\mathbf{s}-\mathbf{z_2}) \\ &= \frac{\rho}{3\varepsilon_0}\left[\begin{pmatrix} x \\ y \\ z-l \end{pmatrix} + \frac{r^3}{(x^2+y^2+z^2+2zl+l^2)^{3/2}}\begin{pmatrix} x \\ y \\ z+l \end{pmatrix}\right].\end{aligned} \tag{8}$$

When our test charge is inside the lower sphere instead, we just switch the sign of l wherever it occurs:

$$E = \frac{\rho}{3\varepsilon_0}\left[\begin{pmatrix} x \\ y \\ z+l \end{pmatrix} + \frac{r^3}{(x^2+y^2+z^2-2zl+l^2)^{3/2}}\begin{pmatrix} x \\ y \\ z-l \end{pmatrix}\right]. \tag{9}$$

When to use each of these three formulae (eqs. (7) to (9)) can be determined by the following table:

Position	Condition 1	Condition 2	Equation to use
Outside both	$\lvert\mathbf{s}-\mathbf{z_1}\rvert > r$	$\lvert\mathbf{s}-\mathbf{z_2}\rvert > r$	eq. (7)
Inside upper	$\lvert\mathbf{s}-\mathbf{z_1}\rvert \leqslant r$	$\lvert\mathbf{s}-\mathbf{z_2}\rvert > r$	eq. (8)
Inside lower	$\lvert\mathbf{s}-\mathbf{z_1}\rvert > r$	$\lvert\mathbf{s}-\mathbf{z_2}\rvert \leqslant r$	eq. (9)

Writing these conditions in terms of scalars, if $|\mathbf{s}-\mathbf{z_1}| > r$ then

$$\sqrt{x^2+y^2+z^2-2zl+l^2} > r$$

and likewise if $|\mathbf{s}-\mathbf{z_2}| > r$ then

$$\sqrt{x^2+y^2+z^2+2zl+l^2} > r.$$

We can express the electric field as a piecewise function in Mathematica:

In[32]:= $\mathtt{Field}[x\_, y\_, z\_] :=$

$$\begin{aligned}&\mathtt{Piecewise}\Big[\Big\{\Big\{\frac{r^3\rho}{3\epsilon}\left(\frac{1}{(x^2+y^2+z^2-2zl+l^2)^{3/2}}\begin{pmatrix} x \\ y \\ z-l \end{pmatrix} + \frac{1}{(x^2+y^2+z^2+2zl+l^2)^{3/2}}\begin{pmatrix} x \\ y \\ z+l \end{pmatrix}\right), \\ &\quad \sqrt{x^2+y^2+z^2-2zl+l^2} > r \land \sqrt{x^2+y^2+z^2+2zl+l^2} > r\Big\}, \\ &\quad \Big\{\frac{\rho}{3\epsilon}\left(\begin{pmatrix} x \\ y \\ z-l \end{pmatrix} + \frac{r^3}{(x^2+y^2+z^2+2zl+l^2)^{3/2}}\begin{pmatrix} x \\ y \\ z+l \end{pmatrix}\right), \\ &\quad \sqrt{x^2+y^2+z^2-2zl+l^2} \leq r \land \sqrt{x^2+y^2+z^2+2zl+l^2} > r\Big\}, \\ &\quad \Big\{\frac{\rho}{3\epsilon}\left(\begin{pmatrix} x \\ y \\ z+l \end{pmatrix} + \frac{r^3}{(x^2+y^2+z^2-2zl+l^2)^{3/2}}\begin{pmatrix} x \\ y \\ z-l \end{pmatrix}\right), \\ &\quad \sqrt{x^2+y^2+z^2-2zl+l^2} > r \land \sqrt{x^2+y^2+z^2+2zl+l^2} \leq r\Big\}\Big\}\Big]\end{aligned}$$

This leads to some rather nice visualisations, as shown in figs. 4 and 5. We set l to 2 m and the radius r to 1 m.

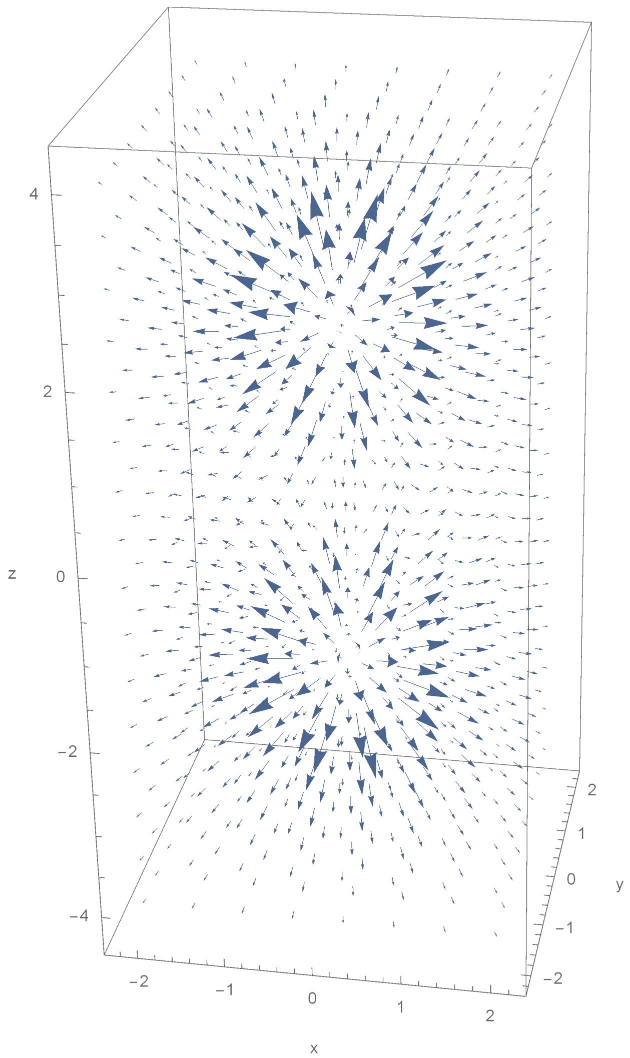


Figure 4: A 3D vector field visualisation of the electric field $\vec{E}$ for the spheres with like charges.

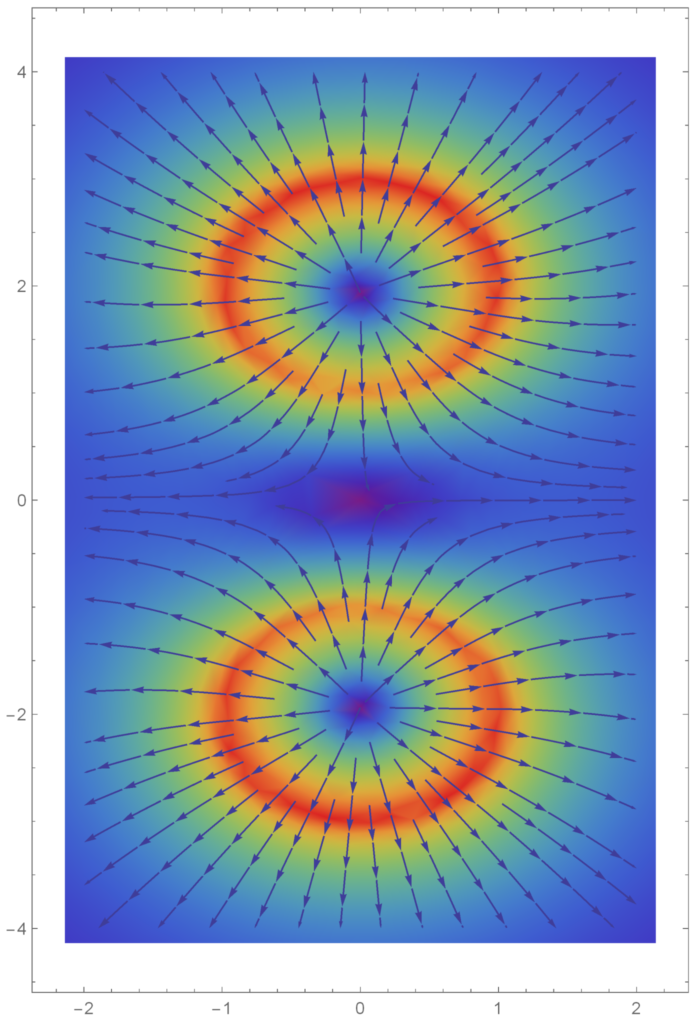


Figure 5: A stream plot overlayed on a density plot in the plane $y = 0$ (for the spheres with like charges).

So far we have assumed the spheres are either both positively charged, or both negatively charged. If we set them to have opposite charges, the visualisation changes. Our piecewise function becomes:

In[85]:= **Field'**[x_, y_, z_] :=

$$\textbf{Piecewise}\Big[\Big\{\Big\{\frac{r^3\rho}{3\epsilon}\left(\frac{1}{(x^2+y^2+z^2-2zl+l^2)^{3/2}}\begin{pmatrix}x\\y\\z-l\end{pmatrix}-\frac{1}{(x^2+y^2+z^2+2zl+l^2)^{3/2}}\begin{pmatrix}x\\y\\z+l\end{pmatrix}\right),$$
$$\sqrt{x^2+y^2+z^2-2zl+l^2}>r\wedge\sqrt{x^2+y^2+z^2+2zl+l^2}>r\Big\},$$
$$\Big\{\frac{\rho}{3\epsilon}\left(\begin{pmatrix}x\\y\\z-l\end{pmatrix}-\frac{r^3}{(x^2+y^2+z^2+2zl+l^2)^{3/2}}\begin{pmatrix}x\\y\\z+l\end{pmatrix}\right),$$
$$\sqrt{x^2+y^2+z^2-2zl+l^2}\le r\wedge\sqrt{x^2+y^2+z^2+2zl+l^2}>r\Big\},$$
$$\Big\{\frac{\rho}{3\epsilon}\left(-\begin{pmatrix}x\\y\\z+l\end{pmatrix}+\frac{r^3}{(x^2+y^2+z^2-2zl+l^2)^{3/2}}\begin{pmatrix}x\\y\\z-l\end{pmatrix}\right),$$
$$\sqrt{x^2+y^2+z^2-2zl+l^2}>r\wedge\sqrt{x^2+y^2+z^2+2zl+l^2}\le r\Big\}\Big\}\Big]$$

Figures 6 to 9 show the field when the two spheres have opposite charges.

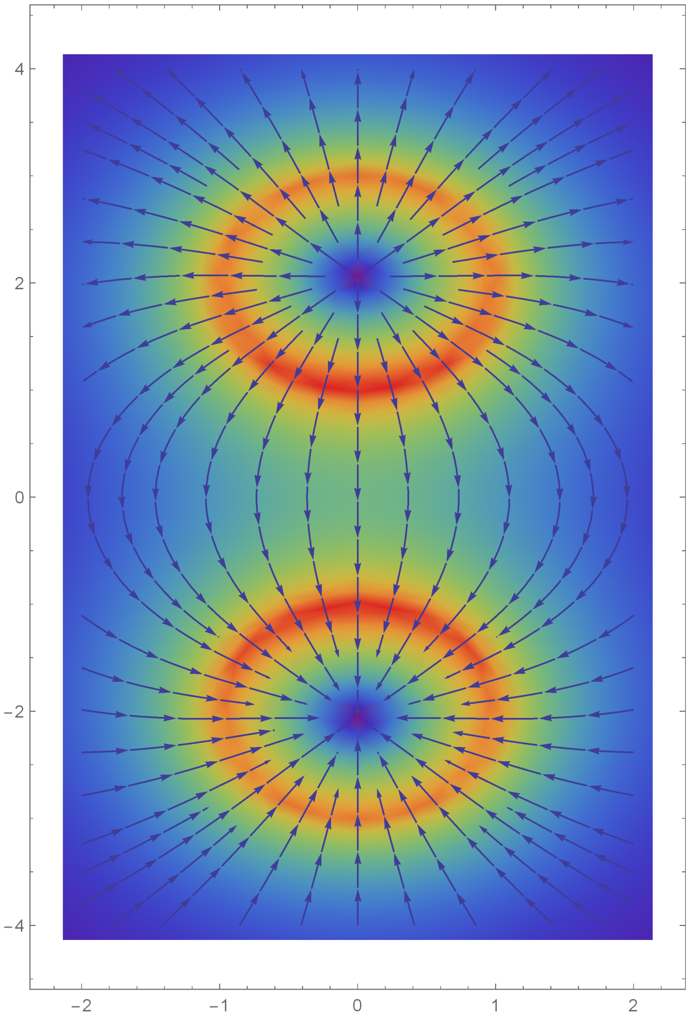


Figure 6: A stream plot and density plot at $y = 0$ for the spheres with opposite charges.

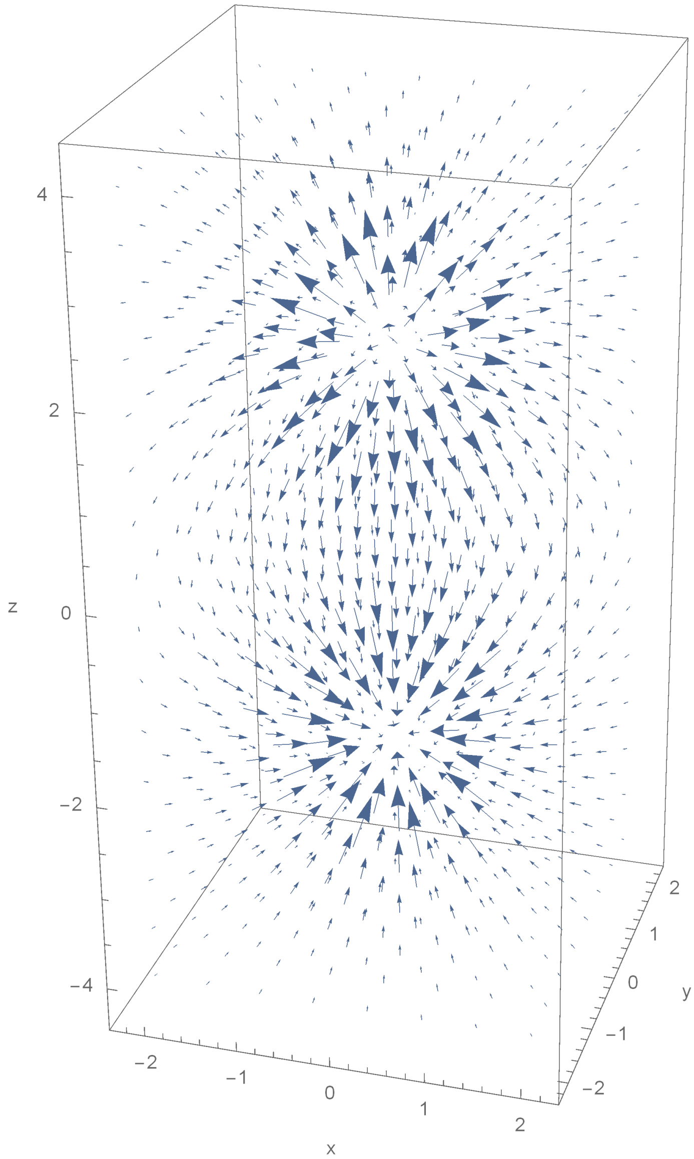


Figure 7: A 3D vector field visualisation of the electric field $\vec{E}$ for the spheres with opposite charges.

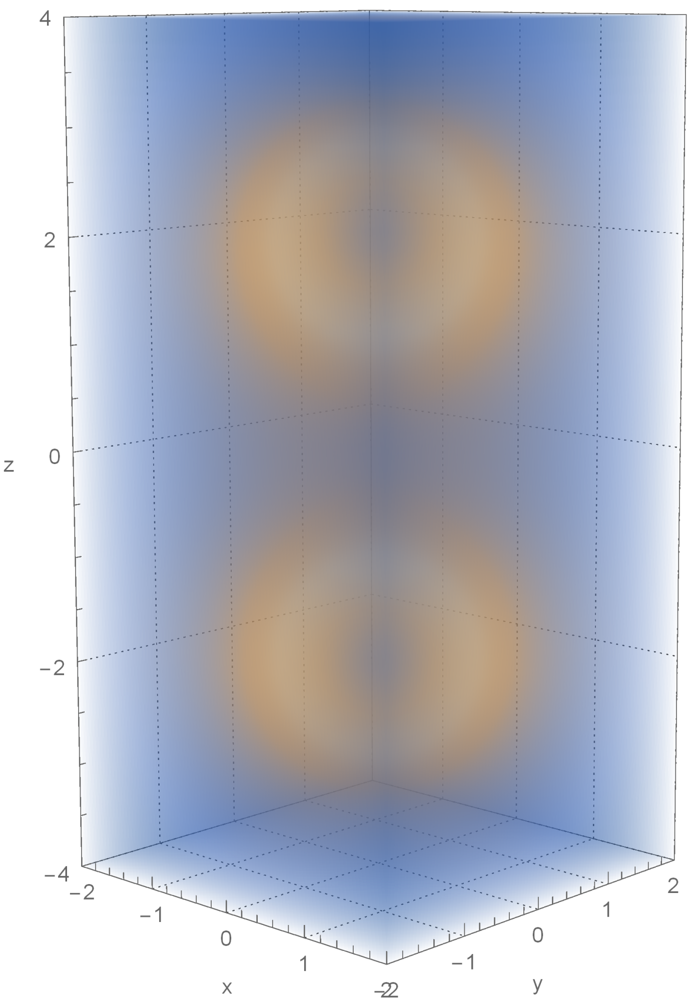


Figure 8: A 3D plot of the electric field magnitude E for the spheres with opposite charges.

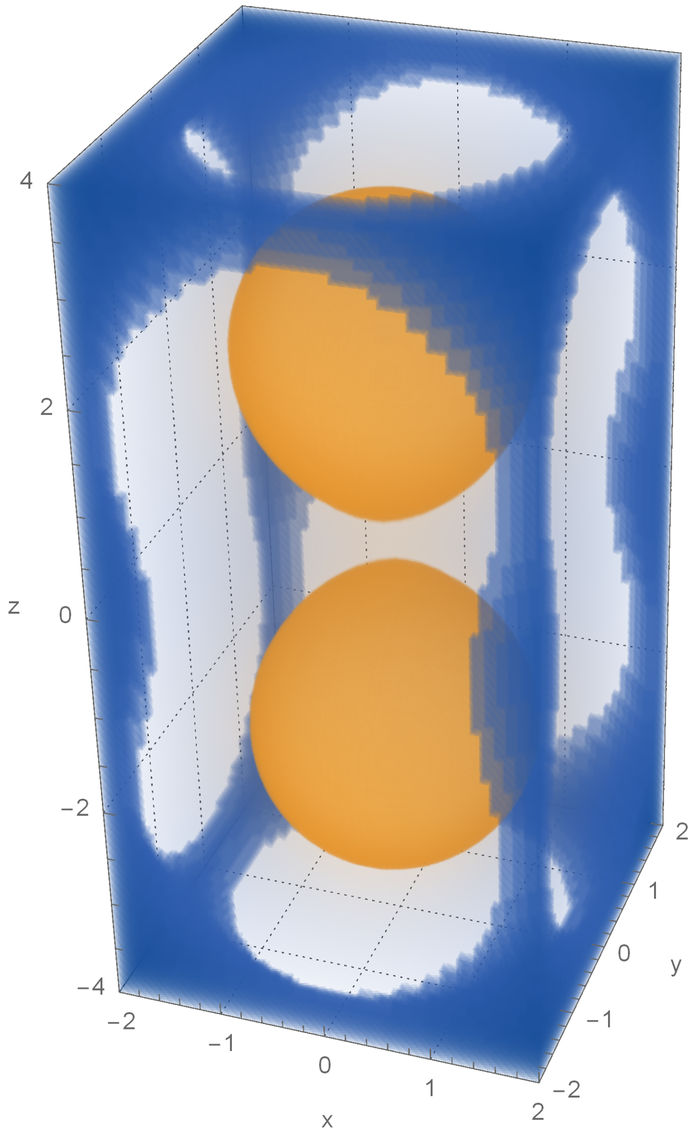


Figure 9: Another 3D plot of the electric field magnitude E for the spheres with opposite charges, this time with the opacity function adjusted to make the spheres more visible.

IV THE FORCE BETWEEN TWO SPHERES OF DIFFERENT VOLUMES

We'll now consider two solid spheres of radius r and R centred at arbitrary positions in Cartesian space $\mathbf{a}$ and $\mathbf{b}$ respectively, such that $|\mathbf{a}-\mathbf{b}| > r+R$. We wish to find the net force each sphere exerts on the other due to their electric fields; we will assume the first sphere is positively charged and the second is negatively charged, both with volume charge density ρ_c.

As shown previously, we can model each sphere as a point charge. The first sphere will have charge

$$q = \rho_c V = \frac{4}{3}\pi r^3 \rho_c \tag{10}$$

and the second will have charge

$$Q = \frac{4}{3}\pi R^3 \rho_c. \tag{11}$$

Therefore, by direct application of Coulomb's law, the force that the first sphere exerts on the second is

$$\begin{aligned}\mathbf{F} &= \frac{\frac{4}{3}\pi r^3 \rho_c \cdot \frac{4}{3}\pi R^3 \rho_c}{4\pi\varepsilon_0 |\mathbf{a}-\mathbf{b}|^3}(\mathbf{a}-\mathbf{b}) \\ &= \frac{4\pi\rho_c^2 r^3 R^3}{9\varepsilon_0 |\mathbf{a}-\mathbf{b}|^3}(\mathbf{a}-\mathbf{b})\end{aligned}$$

and by Newton's third law, the corresponding force from the second sphere on the first is $-\mathbf{F}$.

V GENERALISED MOTION OF TWO FREE-MOVING CHARGED SPHERES

In order to use equations of motion, we must have mass. Let both spheres have volume mass density ρ_m. Then, by analogy with eqs. (10) and (11), the mass of the first sphere is

$$m = \frac{4}{3}\pi r^3 \rho_m$$

and the mass of the second sphere is

$$M = \frac{4}{3}\pi R^3 \rho_m.$$

Therefore, at any given moment, the acceleration of the first sphere is

$$\ddot{\mathbf{a}} = \frac{\mathbf{F}}{m} = \frac{\frac{4\pi\rho_c^2 r^3 R^3}{9\varepsilon_0 |\mathbf{a}-\mathbf{b}|^3}(\mathbf{b}-\mathbf{a})}{\frac{4}{3}\pi r^3 \rho_m} = \frac{\rho_c^2 R^3}{3\varepsilon_0 \rho_m |\mathbf{a}-\mathbf{b}|^3}(\mathbf{b}-\mathbf{a})$$

and the acceleration of the second sphere is

$$\ddot{\mathbf{b}} = -\frac{\mathbf{F}}{M} = -\frac{\frac{4\pi\rho_c^2 r^3 R^3}{9\varepsilon_0 |\mathbf{a}-\mathbf{b}|^3}(\mathbf{a}-\mathbf{b})}{\frac{4}{3}\pi R^3 \rho_m} = \frac{\rho_c^2 r^3}{3\varepsilon_0 \rho_m |\mathbf{a}-\mathbf{b}|^3}(\mathbf{a}-\mathbf{b}).$$

Thus our problem reduces to a pair of coupled second-order non-linear ordinary differential equations. We won't try to solve these, but rather we'll numerically solve them in Mathematica.

Stripping away all constants for now, the following code numerically generates functions for $\mathbf{a}$ and $\mathbf{b}$ (we set their initial velocities to zero and their initial positions to $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ respectively):

In[221]:=
```
Positions =
 NDSolve[{a''[t] == 1/Norm[a[t] - b[t]]^3 (b[t] - a[t]),
   b''[t] == 1/Norm[a[t] - b[t]]^3 (a[t] - b[t]), a[0] == {0, 0, -1},
   b[0] == {0, 0, 1}, a'[0] == {0, 0, 0}, b'[0] == {0, 0, 0}}, {a, b}, {t, 0, 100}]
```

Out[221]=
```
{{a → InterpolatingFunction[ Domain: {{0., 2.22}} Output dimensions: {3} ],
  b → InterpolatingFunction[ Domain: {{0., 2.22}} Output dimensions: {3} ]}}
```

Now we'll add the constants back in (with $\rho_c = \lambda$ and $\rho_m = \mu$). Setting $\lambda = \mu = 1$ and the constant of permittivity $\varepsilon = \frac{1}{3}$ for convenience, we allow control of the radii r and R as well as all of the initial positions and velocities through a 'Manipulate' function. For now we'll plot the z-positions of the two spheres from $t = 0$ to t_max. The code is as follows:

In[247]:=
```
Manipulate[
 PositionFunc =
  NDSolve[{a''[t] == (λ^2 R^3)/(3 ϵ μ Norm[a[t] - b[t]]^3) (b[t] - a[t]),
    b''[t] == (λ^2 r^3)/(3 ϵ μ Norm[a[t] - b[t]]^3) (a[t] - b[t]), a[0] == {sax, say, saz},
    b[0] == {sbx, sby, sbz}, a'[0] == {vax, vay, vaz}, b'[0] == {vbx, vby, vbz}},
   {a, b}, {t, 0, 100}];
 Plot[{Evaluate[a[time][[3]] /. PositionFunc],
   Evaluate[b[time][[3]] /. PositionFunc]}, {time, 0, tmax}], {{r, 1}, 0.1, 10},
 {{R, 1}, 0.1, 10}, {{tmax, 50}, 0.001, 100}, {{sax, 0}, -1, 1}, {{say, 0}, -1, 1},
 {{saz, -1}, -1, 1}, {{sbx, 0}, -1, 1}, {{sby, 0}, -1, 1}, {{sbz, 1}, -1, 1},
 {{vax, 0}, -1, 1}, {{vay, 0}, -1, 1}, {{vaz, 0}, -1, 1}, {{vbx, 0}, -1, 1},
 {{vby, 1}, -1, 1}, {{vbz, 0}, -1, 1}]
```

The resulting manipulate applet is shown in fig. 10.

VI ORBITAL MOTION

We'll now set up a small test charge in orbit around a larger one. Let's make a widget to animate the spheres' motion in three dimensions to aid visualisation.

Based on the previous part, the following code contains an 'Animate' function within a 'Manipulate' function, which allows customised animations based on different starting conditions. I've also included a little plot similar to in fig. 10 to make things easier.

Out[247]=

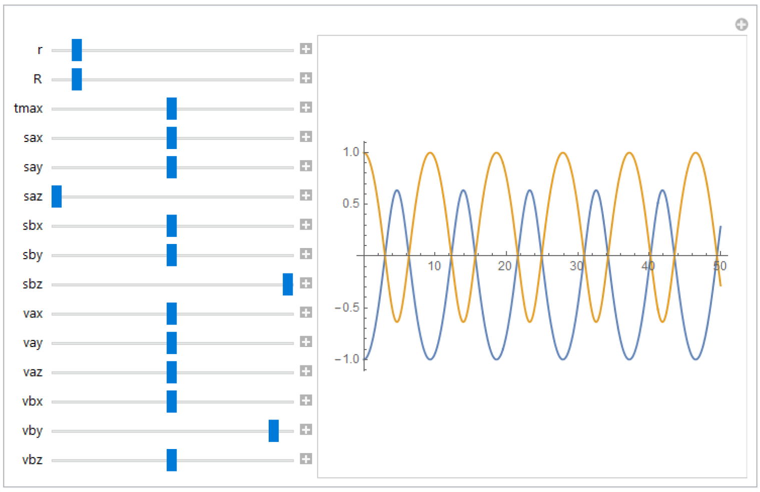


Figure 10: A manipulate function with control over both spheres' radii and starting positions. We already see some nice patterns developing.

In[274]:=
```
Manipulate[
 PositionFunc =
  NDSolve[{a''[t] == (λ^2 R^3)/(3 ϵ μ Norm[a[t] - b[t]]^3) (b[t] - a[t]), b''[t] == (λ^2 r^3)/(3 ϵ μ Norm[a[t] - b[t]]^3) (a[t] - b[t]),
    a[0] == {sax, say, saz}, b[0] == {sbx, sby, sbz}, a'[0] == {vax, vay, vaz}, b'[0] == {vbx, vby, vbz}}, {a, b}, {t, 0, 100}];
 Grid[
  {{Plot[{Evaluate[a[time][[3]] /. PositionFunc], Evaluate[b[time][[3]] /. PositionFunc]}, {time, 0, tmax},
     ImageSize → Small],
    Animate[Graphics3D[{Sphere[Evaluate[a[time] /. PositionFunc], r], Sphere[Evaluate[b[time] /. PositionFunc], R],
       Opacity[0], Sphere[{0, 0, 0}, 2]}, ViewPoint → {1.3, -2.4, 2.}, Boxed → True, FaceGrids → All, ImageSize → Large],
     {time, 0, tmax}]}}], {{r, .5}, 0.01, 5}, {{R, .5}, 0.01, 5}, {{tmax, 50}, 0.001, 100}, {{sax, 0}, -5, 5},
 {{say, 0}, -5, 5}, {{saz, -1}, -5, 5}, {{sbx, 0}, -5, 5}, {{sby, 0}, -5, 5}, {{sbz, 1}, -5, 5}, {{vax, 0}, -5, 5},
 {{vay, 0}, -5, 5}, {{vaz, 0}, -5, 5}, {{vbx, 0}, -5, 5}, {{vby, 1}, -5, 5}, {{vbz, 0}, -5, 5}]
```

We can experiment with all sorts of different radii and sizes and come up with lots of different orbital patterns. I recommend just trying out the attached applet to see the animation working.

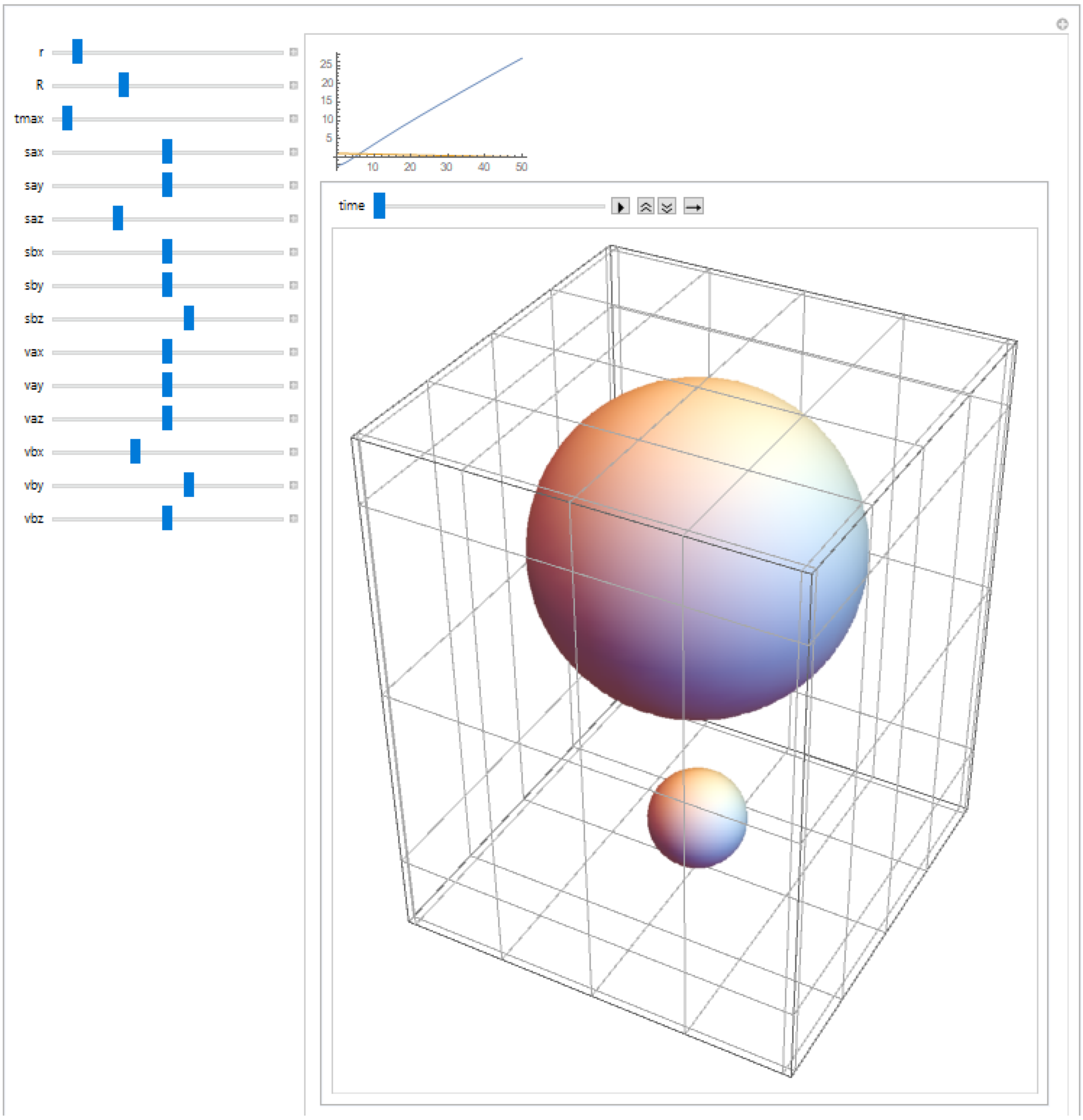


Figure 11: A screenshot of the animation of orbital motion.